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SOLUTIONS OF EXERCISES.

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H. Y. Benedict 326 ; Geo. R. Dean 331, 337, 338, 339, 346 ; Wm. H. Echols 336, 339, 343 ; Angelo Hall 339 ; J. E. Hendricks 325 ; W. W. Johnson 313 ; Artemas Martin 337, 339 ; Frank Morley 332 ; Chas. Puryear 339 ; W. B. Richards 337, 338, 346 ; H. C. Riggs 338 ; Nicolas Schwedton 330 ; W. M. Thornton 339 ; W. O. Whitscarver 346 ; Chas. Yardley 330.

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FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \dots + [\tfrac{1}{2}n(n+1)]^2.$$

[Artemas Martin.]

SOLUTION.

Putting S for the sum sought we have by the method of differences,

$$\begin{aligned} S = n + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 \\ + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} d_3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} d_4 + \dots, \end{aligned}$$

where d_1, d_2, d_3, d_4 , etc. are the first terms of the first, second, third, fourth, etc. orders of differences.

The given series, expanding the terms, is

$$1, \quad 9, \quad 36, \quad 100, \quad 225, \quad 441, \quad 784, \quad 1296, \quad \dots,$$

and the differences are

$$\begin{array}{cccccccc} 8, & 27, & 64, & 125, & 216, & 343, & 512, & \dots, \\ 19, & 37, & 61, & 91, & 127, & 169, & & \dots, \\ 18, & 24, & 30, & 36, & 42, & & & \dots, \\ & 6, & 6, & 6, & 6, & & & \dots, \\ & & 0, & 0, & 0, & & & \dots \end{array}$$

Hence $d_1 = 8, d_2 = 19, d_3 = 18, d_4 = 6, d_5 = 0$. Substituting these values in the expression for S and reducing, we find

$$S = \tfrac{1}{60} n(n+1)(n+2)(3n^2 + 6n + 1).$$

[Charles Yardley.]

331

THE extremities of a diameter of a variable ellipse having fixed foci lie on a fixed hyperbola having the same foci; show that the extremities of the conjugate diameter lie on another hyperbola having the same foci.

[*W. Woolsey Johnson.*]

SOLUTION.

Let the equation of the fixed hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

An ellipse having the same foci may be represented by the equation

$$\frac{x^2}{a^2 + \lambda^2} + \frac{y^2}{\lambda^2 - b^2} = 1.$$

For their points of intersection,

$$x^2 = \frac{(a^2 + \lambda^2)a^2}{a^2 + b^2}, \quad y^2 = \frac{b^2(\lambda^2 - b^2)}{a^2 + b^2}.$$

If x', y' be the coordinates of the extremity of the conjugate diameter,

$$x'^2 = y^2 \cdot \frac{a^2 + \lambda^2}{\lambda^2 - b^2}, \quad y'^2 = x^2 \cdot \frac{\lambda^2 - b^2}{a^2 + \lambda^2};$$

whence $\frac{x'^2}{b^2} - \frac{y'^2}{a^2} = 1$, a confocal hyperbola.

[*Geo. R. Dean.*]

337

REQUIRED the locus of the foot of the perpendicular from the centre of an ellipse upon the common chord of the ellipse and circle of curvature.

[*Artemas Martin.*]

SOLUTION.

The equation to the chord of curvature, as given in Salmon's Conics, p. 229, Ex. 4, is

$$\frac{x}{a} \cos \alpha - \frac{y}{b} \sin \alpha = \cos 2\alpha. \quad (1)$$

The perpendicular from the centre on this is

$$x = -\frac{b}{a} y \cot \alpha;$$

whence

$$\tan \alpha = -\frac{by}{ax}.$$

Substituting in (1) the values of $\cos \alpha$, $\sin \alpha$, $\cos 2\alpha$ obtained from this, we have

$$\frac{x^2}{\sqrt{a^2x^2 + b^2y^2}} + \frac{y^2}{\sqrt{a^2x^2 + b^2y^2}} = \frac{a^2x^2 - b^2y^2}{a^2x^2 + b^2y^2},$$

or clearing of fractions and radicals,

$$(x^2 + y^2)^2 (a^2x^2 + b^2y^2) = (a^2x^2 - b^2y^2)^2.$$

[W. B. Richards.]

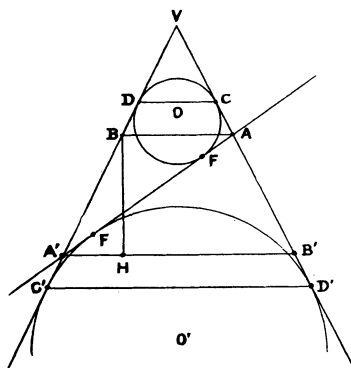
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PROVE synthetically that the eccentricity of a conic section is equal to the sine of the angle which the cutting plane makes with the base of the cone, divided by the sine of the angle which an element of the cone makes with the base.

[H. B. Newson.]

SOLUTION.

Let $VC'D'$ be the axial section of the cone and AA' the cutting plane.



Let O and O' be spheres inscribed in the cone touching the plane AA' at the points F and F' , and let CD and $C'D'$ be the circles where the spheres O and O' touch the cone.

Pass the planes AB and $A'B'$ through A and A' parallel to the planes of the circles CD and $C'D'$. Draw the line BH perpendicular to $A'B'$. Let the angle $AA'B$ be α , and $BA'B'$ be θ .

It can be easily proved that F and F' are the foci of the conic, and that the line DC' is equal to the major axis AA' (see *Salmon's*

Conic Sections, Art. 367); whence

$$DC' = AA'. \quad (1)$$

But

$$AC = AF, \quad (2)$$

and

$$A'C' = A'F', \quad (3)$$

since they are tangents to a sphere from the same point.

Subtracting (2) and (3) from (1), we obtain

$$DA' - CA = FF';$$

or, since

$$CA = DB,$$

$$A'B = FF'.$$

Now

$$e = \frac{FF'}{AA'} = \frac{BA'}{AA'},$$

and

$$BA' = \frac{BH}{\sin \theta}, \quad AA' = \frac{BH}{\sin \alpha};$$

whence

$$e = \frac{\sin \alpha}{\sin \theta}.$$

Q. E. D.

[H. C. Riggs.]

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SHOW that

$$\sum_{m=0}^{m=n} (-1)^m \frac{2^{2m-1} - 1}{(2n - 2m + 1)!} \frac{B_m}{(2m)!} = 0,$$

where B_m represents Bernoulli's number.

[W. H. Echols.]

SOLUTION.

We know that

$$\frac{x}{e^x + 1} = \frac{x}{2} - (2^2 - 1) \frac{B_1}{2!} x^2 + (2^4 - 1) \frac{B_2}{4!} x^4 - (2^6 - 1) \frac{B_3}{6!} x^6 + \dots,$$

and

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1}{2!} x^2 - \frac{B_2}{4!} x^4 + \frac{B_3}{6!} x^6 - \dots$$

Adding these, we obtain

$$\frac{xe^x}{e^{2x} - 1} = \frac{1}{2} - (2^1 - 1) \frac{B_1}{2!} x^2 + (2^3 - 1) \frac{B_2}{4!} x^4 - (2^5 - 1) \frac{B_3}{6!} x^6 + \dots$$

But we know that

$$\frac{e^{2x} - 1}{2xe^x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \dots$$

Multiplying these two last series together, since they are absolutely convergent, we obtain

$$\begin{aligned} 0 = & \left[-\frac{2^1 - 1}{2!} B_1 + \frac{1}{2} \frac{1}{3!} \right] x^2 + \left[\frac{2^3 - 1}{4!} B_2 - \frac{2^1 - 1}{3!} \frac{B_1}{2!} + \frac{1}{2} \frac{1}{5!} \right] x^4 + \dots \\ & + x^{2n} \sum_{m=0}^{m=n} (-1)^m \frac{2^{2m-1} - 1}{(2n - 2m + 1)!} \frac{B_m}{(2m)!} + \dots, \end{aligned}$$

which establishes the formula.

[W. H. Echols.]